A bilinear Bogolyubov-Ruzsa lemma with polylogarithmic bounds

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Abstract: The Bogolyubov-Ruzsa lemma, in particular the quantitative bound obtained by Sanders, plays a central role in obtaining effective bounds for the $U^3$ inverse theorem for the Gowers norms. Recently, Gowers and Milićević applied a bilinear Bogolyubov-Ruzsa lemma as part of a proof of the $U^4$ inverse theorem with effective bounds. The goal of this note is to obtain a quantitative bound for the bilinear Bogolyubov-Ruzsa lemma which is similar to that obtained by Sanders for the Bogolyubov-Ruzsa lemma.

We show that if a set $A \subset \mathbb{F}^n \times \mathbb{F}^n$ has density $\alpha$, then after a constant number of horizontal and vertical sums, the set $A$ contains a bilinear structure of codimension $r = \log^{O(1)} \alpha^{-1}$. This improves the result of Gowers and Milićević, who obtained a similar statement with a weaker bound of $r = \exp(\exp(\log^{O(1)} \alpha^{-1}))$, and by Bienvenu and Lê, who obtained $r = \exp(\exp(\exp(\log^{O(1)} \alpha^{-1})))$.

Key words and phrases: Additive combinatorics, Bogolyubov-Ruzsa lemma, bilinear set.

1 Introduction

One of the key ingredients in the proof of the quantitative inverse theorem for the Gowers $U^3$ norm over finite fields, due to Green and Tao [GT08] and Samorodnitsky [Sam07], is an inverse theorem concerning the structure of sumsets. The best known result in this direction is the quantitatively improved Bogolyubov-Ruzsa lemma due to Sanders [San12a]. Before introducing it, we fix some common notation. We assume that $\mathbb{F} = \mathbb{F}_p$ is a prime field, where $p$ is a fixed constant, and suppress the exact dependence on $p$ in the bounds. Given a subset $A \subset \mathbb{F}^n$ its density is $\alpha = |A|/|\mathbb{F}|^n$. The subset of $A$ is $2A = A + A = \{a + a' : a, a' \in A\}$, and its difference set is $A - A = \{a - a' : a, a' \in A\}$.  

**Theorem 1.1.** ([San12a]) Let $A \subset \mathbb{F}^n$ be a subset of density $\alpha$. Then there exists a subspace $V$ of $\mathbb{F}^n$ of codimension $O(\log^4 \alpha^{-1})$ such that $V \subset 2A - 2A$.

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In fact, the link between the $U^3$ inverse theorem and inverse sumset results runs deeper. It was shown in [GT10, Lov12] that a $U^3$ inverse theorem with (to date conjectural) polynomial bounds is equivalent to the polynomial Freiman-Ruzsa conjecture, one of the central open problems in additive combinatorics. Given this, one cannot help but wonder whether there is a more general inverse sumset phenomenon that would naturally correspond to quantitative inverse theorems for the $U^k$ norms. In a recent breakthrough, Gowers and Miličević [GM17b] showed that this is indeed the case, at least for the $U^4$ norm. They used a bilinear generalization of Theorem 1.1 to obtain a quantitative $U^4$ inverse theorem.

To be able to explain this result we need to introduce some notation. Let $A \subset \mathbb{F}^n \times \mathbb{F}^n$. Define two operators, capturing subtraction on horizontal and vertical fibers as follows:

$$
\phi_h(A) := \{(x_1 - x_2, y) : (x_1, y), (x_2, y) \in A\},
\phi_v(A) := \{(x, y_1 - y_2) : (x, y_1), (x, y_2) \in A\}.
$$

Given a word $w \in \{h, v\}^k$ define $\phi_w = \phi_{w_1} \circ \ldots \circ \phi_{w_k}$ to be their composition. A bilinear variety $B \subset \mathbb{F}^n \times \mathbb{F}^n$ of codimension $r = r_1 + r_2 + r_3$ is a set defined as follows:

$$
B = \{(x, y) \in V \times W : b_1(x, y) = \ldots = b_{r_3}(x, y) = 0\},
$$

where $V, W \subset \mathbb{F}^n$ are subspaces of codimension $r_1, r_2,$ respectively, and $b_1, \ldots, b_{r_3} : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$ are bilinear forms.

Gowers and Miličević [GM17a] and independently Bienvenu and Lê [BL17] proved the following, although [BL17] obtained a weaker bound of $r = \exp(\exp(\exp(\log^{O(1)} 1/\alpha^{-1})))$.

**Theorem 1.2 ([GM17a, BL17]).** Let $A \subset \mathbb{F}^n \times \mathbb{F}^n$ be of density $\alpha$ and let $w = hvvhvvh$. Then there exists a bilinear variety $B \subset \phi_w(A)$ of codimension $r = \exp(\exp(\log^{O(1)} 1/\alpha^{-1})))$.

To be precise, it was not Theorem 1.2 directly but a more analytic variant of it that was used (combined with many other ideas) to prove the $U^4$ inverse theorem in [GM17b]. However, we will not discuss this analytic variant here.

The purpose of this note is to improve the bound in Theorem 1.2 to $r = \log^{O(1)} 1/\alpha^{-1}$, as was conjectured in [BL17]. Our proof is arguably simpler and is obtained only by invoking Theorem 1.1 a few times, without doing any extra Fourier analysis. The motivation behind this work — other than obtaining a bound that matches the linear case — is to employ this result in a more algebraic framework to obtain a modular and simpler proof of a $U^4$ inverse theorem. Moreover, Theorem 1.3 was recently used by Bienvenu and Lê [BL19] to obtain upper bounds on the correlation of the Möbius function over functions fields with quadratic phases.

One more remark before stating the result is that Theorem 1.2 generalizes Theorem 1.1 because given a set $A \subset \mathbb{F}^n$, one can apply Theorem 1.2 to the set $A' = \mathbb{F}^n \times A$ and find $\{x\} \times V \subset \phi_{w}(A')$, where $x$ is arbitrary, and $V$ a subspace of codimension $3r$. This implies $V \subset 2A - 2A$.

**Theorem 1.3 (Main theorem).** Let $A \subset \mathbb{F}^n \times \mathbb{F}^n$ be of density $\alpha$ and let $w = hvhvvhvvh$. Then there exists a bilinear variety $B \subset \phi_{w}(A)$ of codimension $r = O(\log^{80} 1/\alpha^{-1})$.

Note that the choice of the word $w$ in Theorem 1.3 is $w = hvhvvhvvh$, which is slightly longer than the word $hhvvhh$ in Theorem 1.2. However, for applications this usually does not matter and any constant length $w$ would do the job. In fact, allowing $w$ to be longer is what enables us to obtain a result with a stronger bound.
1.1 A robust analog of Theorem 1.3

Returning to the theorem of Sanders, there is a more powerful variant of Theorem 1.1 which guarantees that $V$ enjoys a stronger property than just being a subset of $2A - 2A$. The stronger property is that every element $y \in V$ can be written in many ways as $y = a_1 + a_2 - a_3 - a_4$, with $a_1, a_2, a_3, a_4 \in A$. This stronger property of $V$ has a number of applications, for example to upper bounds for Roth’s theorem in four variables. We refer the reader to [SS16], where Theorem 3.2 is obtained from Theorem 1.1 and the aforementioned application is given.

Theorem 1.4 ([San12a, SS16]). Let $A \subset \mathbb{F}_n$ be a subset of density $\alpha$. Then there exists a subspace $V \subset 2A - 2A$ of codimension $O(\log^4 \alpha^{-1})$ such that the following holds. Every $y \in V$ can be expressed as $y = a_1 + a_2 - a_3 - a_4$ with $a_1, a_2, a_3, a_4 \in A$ in at least $\alpha^{O(1)}|\mathbb{F}|^{3n}$ many ways.

In Section 3 we also prove a bilinear version of Theorem 1.4 by slightly modifying the proof of Theorem 1.3. To explain it, we need just a bit more notation.

Fix an arbitrary $(x, y) \in \mathbb{F}^n \times \mathbb{F}^n$, and note that $(x, y)$ can be written as $(x, y) = \phi_v((x + x_1, y), (x_1, y))$ for any $x_1 \in \mathbb{F}^n$. Moreover, for any fixed $x_1$, each of the points $(x + x_1, y), (x_1, y)$ can be written as $(x + x_1, y) = \phi_v((x + x_1, y + y_1), (x_1, y_1))$ and $(x_1, y) = \phi_v((x_1, y + y_2), (x_1, y_2))$ for arbitrary $y_1, y_2 \in \mathbb{F}^n$. So overall, the point $(x, y)$ can be written using the operation $\phi_{vh}$ in exactly $|\mathbb{F}|^3$ many ways, namely, the total number of two-dimensional parallelograms $(x + x_1, y + y_1), (x + x_1, y_1), (x_1, y + y_2), (x_1, y_2)$, where $(x, y)$ is fixed. More generally, for an arbitrary word $w \in \{h, v\}^k$, the point $(x, y)$ can be written using the operation $\phi_w$ in exactly $|\mathbb{F}|^{2k-1}$ many ways.

Given a set $A \subset \mathbb{F}^n \times \mathbb{F}^n$ and a word $w \in \{h, v\}^k$, we define $\phi_w(A)$ to be the set of all elements $(x, y) \in \mathbb{F}^n \times \mathbb{F}^n$ that can be obtained in at least $\varepsilon |\mathbb{F}|^{2k-1}$ many ways by applying the operation $\phi_w(A)$.

The following is an extension of Theorem 1.3 similar in spirit to Theorem 1.4.

Theorem 1.5. Let $A \subset \mathbb{F}^n \times \mathbb{F}^n$ be of density $\alpha$ and let $w = hvhvvvh$ and $\varepsilon = \exp(-O(\log^{20} \alpha^{-1}))$. Then there exists a bilinear variety $B \subset \phi_w(A)$ of codimension $r = O(\log^{80} \alpha^{-1})$.

As a final comment, we remark that if one keeps track of the dependence on the size of the field throughout the proofs, then the bound in Theorem 1.3 and Theorem 1.5 is $r = O(\log^{80} \alpha^{-1} \cdot \log^{O(1)} |\mathbb{F}|)$.

Paper organization. We prove Theorem 1.3 in Section 2 and Theorem 1.5 in Section 3.

2 Proof of Theorem 1.3

We prove Theorem 1.3 in six steps, which correspond to applying the chain of operators $\phi_h \circ \phi_v \circ \phi_h \circ \phi_v \circ \phi_{hv} \circ \phi_{hh}$ to $A$. In the proof, we invoke Theorem 1.1 (or Theorem 1.4, or the Freiman-Ruzsa theorem, which is a corollary of Theorem 1.1), four times in total, in Steps 1, 2, 4, and 5.

We will assume that $A \subset \mathbb{F}^m \times \mathbb{F}^n$, where initially $m = n$ but where throughout the proof we update $m, n$ independently when we restrict $x$ or $y$ to large subspaces. It also helps readability, as we will always have that $x$ and related sets or subspaces are in $\mathbb{F}^m$, while $y$ and related sets or subspaces are in $\mathbb{F}^n$.

We use three variables $r_1, r_2, r_3$ that hold the total number of linear forms on $x$, linear forms on $y$, and bilinear forms on $(x, y)$ that are being fixed throughout the proof, respectively. Initially, $r_1 = r_2 = r_3 = 0$, but their values will be updated as we go along and at the end $r = r_1 + r_2 + r_3$ will be the codimension of the final bilinear variety.
We pause for a moment to introduce one more useful piece of notation. Recall that an affine map
where each $V \subset \mathbb{P}^n$. Define $A^1 := \phi_{ih}(A)$, so that
$$A^1 = \bigcup_{y \in \mathbb{P}^n} (2A_y - 2A_y) \times \{y\}.$$  

Let $\alpha_y$ denote the density of $A_y$. By Theorem 1.1, there exists a linear subspace $V'_y \subset 2A_y - 2A_y$ of codimension $O(\log^4 \alpha_y^{-1})$. Let $S := \{y : \alpha_y \geq \alpha/2\}$, where by averaging $S$ has density $\geq \alpha/2$. Note that for every $y \in S$ the codimension of each $V'_y$ is $O(\log^4 \alpha^{-1})$. We have
$$B^1 := \bigcup_{y \in S} V'_y \times \{y\} \subset A^1.$$  

**Step 2.** Consider $A^2 := \phi_{nv}(B^1)$. It satisfies
$$A^2 = \bigcup_{y_1,y_2,y_3,y_4 \in S} (V'_{y_1} \cap V'_{y_2} \cap V'_{y_3} \cap V'_{y_4}) \times \{y_1 + y_2 - y_3 - y_4\}.$$  

By Theorem 1.1, there is a subspace $W' \subset 2S - 2S$ of codimension $O(\log^4 \alpha^{-1})$. Note that the codimension of $W'$, as well as the codimension of each $V'_{y_1} \cap V'_{y_2} \cap V'_{y_3} \cap V'_{y_4}$, is at most $O(\log^4 \alpha^{-1})$. We thus have
$$B^2 := \bigcup_{y \in W'} V_y \times \{y\},$$  
where $V_y = V'_{y_1} \cap V'_{y_2} \cap V'_{y_3} \cap V'_{y_4}$ for some $y_1,y_2,y_3,y_4 \in S$ which satisfy $y = y_1 + y_2 - y_3 - y_4$.

Update $r_2 := \text{codim}(W')$, where we restrict $y \in W'$. To simplify the notation, identify $W' \equiv \mathbb{P}^{n - \text{codim}(W')}$ and update $n := n - \text{codim}(W')$. Thus we assume from now that
$$B^2 := \bigcup_{y \in \mathbb{P}^n} V_y \times \{y\},$$  
where each $V_y$ has codimension $d = O(\log^4 \alpha^{-1})$.

**Step 3.** Consider $A^3 := \phi_{v}(B^2)$. It satisfies
$$A^3 = \bigcup_{y,z \in \mathbb{P}^n} (V_z \cap V_{y+z}) \times \{y\}.$$  

**Step 4.** Consider $A^4 := \phi_{h}(A^3)$. It satisfies
$$A^4 = \bigcup_{y,z,w \in \mathbb{P}^n} ((V_z \cap V_{y+z}) + (V_w \cap V_{y+w})) \times \{y\}.$$  

Define $U_y := V_y^1$, so that $\text{dim}(U_y) = d$ and
$$A^4 = \bigcup_{y,z,w \in \mathbb{P}^n} ((U_z + U_{y+z}) \cap (U_w + U_{y+w}))^1 \times \{y\}.$$  

We pause for a moment to introduce one more useful piece of notation. Recall that an affine map $L : \mathbb{P}^n \to \mathbb{P}^m$ is of the form $L(y) = My + b$ where $M \in \mathbb{F}^{m \times n}, b \in \mathbb{F}^m$. Given a set of affine maps $\mathcal{L} = \{L_i : \mathbb{P}^n \to \mathbb{P}^m, i \in [k]\}$
and \( y \in \mathbb{F}^n \), let \( \mathcal{L}(y) = \{L_1(y), \ldots, L_k(y)\} \subset \mathbb{F}^m \), and let \( \overline{\mathcal{L}} \) denote the linear span of \( \mathcal{L} \). Our goal in this step is to find a small family of affine maps \( \mathcal{L} \) with \( |\mathcal{L}| = O(d) \), and a fixed choice of \( z, w \), so that

\[
\Pr_{y \in \mathbb{F}^n} \left[ (U_z + U_{y+z}) \cap (U_w + U_{y+w}) \subset \overline{\mathcal{L}}(y) \right] \geq 1, \tag{1}
\]
as this will give us a dense set \( T \subset \mathbb{F}^n \) so that

\[
\bigcup_{y \in T} \mathcal{L}(y)^\perp \times \{y\} \subset A^d.
\]

We now explain how to get Equation (1). For every \( a \in \mathbb{F}^n \), let \( \mathcal{L}_a \) be a collection of affine maps where initially \( \mathcal{L}_a = \{0\} \) for all \( a \)'s. We keep adding affine maps to some of the \( \mathcal{L}_a \)'s, while always maintaining \( |\mathcal{L}_a| \leq 2^d \) for all \( a \in \mathbb{F}^n \), until we satisfy

\[
\Pr_{y, z, w \in \mathbb{F}^n} \left[ (U_z + U_{y+z}) \cap (U_w + U_{y+w}) \subset \mathcal{L}_a(z) + \mathcal{L}_a(y+z) + \mathcal{L}_a(w) + \mathcal{L}_a(y+w) \right] \geq \frac{1}{2}, \tag{2}
\]
and then we will pick some popular affine maps \( \mathcal{L} \subset \bigcup_{a \in \mathbb{F}^n} \mathcal{L}_a \) with \( |\mathcal{L}| = O(d) \) that will give us Equation (1). For now, we show how to get Equation (2). We need the following lemma.

**Lemma 2.1.** For each \( y \in \mathbb{F}^n \), let \( U_y \subset \mathbb{F}^m \) be a subspace of dimension \( d \). Assume that

\[
\Pr_{y, z, w \in \mathbb{F}^n} \left[ (U_z + U_{y+z}) \cap (U_w + U_{y+w}) \subset \mathcal{L}_a(z) + \mathcal{L}_a(y+z) + \mathcal{L}_a(w) + \mathcal{L}_a(y+w) \right] \leq \frac{1}{2}.
\]

Then there exists an affine function \( L : \mathbb{F}^n \rightarrow \mathbb{F}^m \) such that

\[
\Pr_{y \in \mathbb{F}^n} \left[ L(y) \in U_y \setminus \mathcal{L}_a(y) \right] \geq \exp(-O(d^4)).
\]

In the following we prove Lemma 2.1. We will use a modified version of a functional version of the Freiman-Ruzsa theorem, with the quasi-polynomial bounds obtained by Sanders [San12a]. We first recall the standard version. For details of how it is derived from Theorem 1.1 we refer the reader to [Gre05]. In fact, in this case the bound can be slightly improved. The reader is referred to [San12b].

**Theorem 2.2.** (Freiman-Ruzsa theorem; functional version). Let \( f : \mathbb{F}^n \rightarrow \mathbb{F}^m \) be a function. Suppose that

\[
\Pr_{y, z, z' \in \mathbb{F}^n} \left[ f(y+z) - f(z) = f(y+z') - f(z') \right] \geq \alpha.
\]

Then there exists an affine map \( L : \mathbb{F}^n \rightarrow \mathbb{F}^m \) such that

\[
|\{z \in \mathbb{F}^n : L(z) = f(z)\}| \geq \exp(-O(\log^4(\alpha^{-1})))|\mathbb{F}^n|.
\]

Now, this result may be strengthened as follows.

**Lemma 2.3.** Let \( f : \mathbb{F}^n \rightarrow \mathbb{F}^m \) be a function and \( Z \subset \mathbb{F}^n \) with \( |Z| \geq \alpha|\mathbb{F}^n| \). Suppose that

\[
\Pr_{y \in \mathbb{F}^n, z, z' \in Z} \left[ f(y+z) - f(z) = f(y+z') - f(z') \right] \geq \alpha.
\]

Then there exists an affine map \( L : \mathbb{F}^n \rightarrow \mathbb{F}^m \) such that

\[
|\{z \in Z : L(z) = f(z)\}| \geq \exp(-O(\log^4(\alpha^{-1})))|\mathbb{F}^n|.
\]
Proof. Let $\Gamma = \{(x, f(x)) : x \in \mathbb{F}^n\}$ and $\Gamma' = \{(x, f(x)) : x \in \mathbb{Z}\}$. The additive energy $E(\Gamma, \Gamma')$ is defined as

$$E(\Gamma, \Gamma') = |\{(a, b, c, d) : a - b = c - d, a, c \in \Gamma, b, d \in \Gamma'\}|$$

and satisfies

$$E(\Gamma, \Gamma') \geq \alpha^{O(1)}|\Gamma|^3.$$

Using the Cauchy-Schwarz inequality for additive energy (see Corollary 2.10 in [TV06]), we have

$$E(\Gamma, \Gamma') \leq \sqrt{E(\Gamma, \Gamma') \cdot E(\Gamma', \Gamma')}.$$

Using the fact that $|\Gamma'| \geq \alpha|\Gamma|$, we get that $E(\Gamma', \Gamma') \geq \alpha^{O(1)}|\Gamma|^3$. Let $M \geq m$ be large enough, and define a function $f' : \mathbb{F}^n \to \mathbb{F}^M$ by setting $f'(z) = f(z)$ if $z \in \mathbb{Z}$, and otherwise $f$ takes random values in $\mathbb{F}^M$. Apply Theorem 2.2 to $f'$. The linear function $L$ thus obtained has to necessarily agree with $f'$ (and hence with $f$) on a subset $Z' \subset Z$ of the claimed density.

\[ \Box \]

Remark 2.4. Note that using a bootstrapping argument due to Konyagin, the bound $\exp(-O(\log^4(\alpha^{-1})'])) |\mathbb{F}^n|$ in Theorem 2.2 can be improved to $\exp(-O(\log^{3+o(1)}(\alpha^{-1})))) |\mathbb{F}^n|$ (see Theorem 12.5 in [San12b]). Here, we have used the exponent 4 instead of 3 for aesthetic reasons. Using Lemma 2.3 with exponent $3 + o(1)$ in what follows would result in a final bound of $\log^{63+o(1)} \alpha^{-1}$ instead of $\log^{80} \alpha^{-1}$ in Theorem 1.3 and Theorem 1.5.

Now we may return to the proof of Lemma 2.1.

Proof of Lemma 2.1. Consider a choice of $y, w, z$ for which

$$(U_{y+z} + U_z) \cap (U_{y+w} + U_w) \not\subset \mathcal{I}_{y+z}(y + z) + \mathcal{I}_z(z) + \mathcal{I}_{y+w}(y + w) + \mathcal{I}_w(w).$$

This directly implies that there is an ordered quadruple $(a, b, c, d)$ so that $a \in U_{y+z}, b \in U_z, c \in U_{y+w}, d \in U_w$ with $a - b = c - d \neq 0$ and

$$\left( [a \notin \mathcal{I}_{y+z}(y + z)] \text{ OR } [b \notin \mathcal{I}_z(z)] \right) \text{ AND } \left( [c \notin \mathcal{I}_{y+w}(y + w)] \text{ OR } [d \notin \mathcal{I}_w(w)] \right).$$

Consider all the possible solutions of the above formula, namely:

- $[a \notin \mathcal{I}_{y+z}(y + z)] \text{ AND } [c \notin \mathcal{I}_{y+w}(y + w)]$
- $[b \notin \mathcal{I}_z(z)] \text{ AND } [c \notin \mathcal{I}_{y+w}(y + w)]$
- $[a \notin \mathcal{I}_{y+z}(y + z)] \text{ AND } [d \notin \mathcal{I}_w(w)]$
- $[b \notin \mathcal{I}_z(z)] \text{ AND } [d \notin \mathcal{I}_w(w)]$

One of these cases occur for at least 1/4 of the choices of $y, w, z$; assume without loss of generality that it is the last one. The other cases are analogous.

Next, sample a random function $f : \mathbb{F}^n \to \mathbb{F}^m$ by picking $f(x) \in U_x$ uniformly and independently for each $x \in \mathbb{F}^n$. Note that the quadruple $a, b, c, d$ depends on $y, w, z$, and that for each such choice

$$\Pr[f(y + z) = a, f(z) = b, f(y + w) = c, f(w) = d] \geq |\mathbb{F}|^{-4d}.$$
Note that when this event happens, by construction we have \( f(y + z) - f(z) = f(y + w) - f(w). \) Combining this with the assumption of the lemma, we get

\[
\Pr_{y, z, w \in \mathbb{F}^n} \left[ f(y + z) - f(z) = f(y + w) - f(w), (y + z) \in U_z \setminus U_c(z), (y + w) \in U_w \setminus U_c(w) \right] \geq \frac{1}{2} \cdot \frac{1}{4} \cdot \left| \mathbb{F} \right|^{-4d}.
\]

Fix \( f \) where the above bound holds. Let \( Z = \{ z \colon f(z) \in U_c(z) \}. \) Then, supposing the dependence on the size of the field, we have \( |Z| \geq \exp(-O(d))|\mathbb{F}|^n \) and

\[
\Pr_{y \in \mathbb{F}^n, z, w \in Z} \left[ f(y + z) - f(z) = f(y + w) - f(w) \right] \geq \exp(-O(d)).
\]

By Lemma 2.3, there exists an affine map \( L : \mathbb{F}^n \to \mathbb{F}^m \) and a set \( Z' \subset Z \) with \( |Z'| \geq \exp(-O(d^4))|\mathbb{F}^n| \) such that for all \( z' \in Z', f(z') = L(z') \) and hence \( L(z') \in U_{c'} \setminus U_{c'(z')} \).

Next, we proceed as follows. As long as Equation (2) is satisfied, apply Lemma 2.1 to find an affine map \( L : \mathbb{F}^n \to \mathbb{F}^m \). For every \( x \) that satisfies \( L(x) \in U_z \setminus U_c(x) \), add the map \( L \) to \( \mathcal{L} \). This process needs to stop after \( t = \exp(O(d^5)) \) many steps. Let \( L_1, \ldots, L_4 : \mathbb{F}^n \to \mathbb{F}^m \) be the affine maps obtained in this process. Using this notation, set \( \mathcal{L}' = \cup_{x \in \mathbb{F}^n} \mathcal{L}_x \). For every subspace \( U_z \), there is a set \( \mathcal{L}'_z \subset \mathcal{L}' \) of size \( |\mathcal{L}'_z| \leq d \) such that

\[
\mathcal{L}_c(x) \subset \mathcal{L}'_z(x).
\]

This implies that

\[
\Pr_{y, z, w \in \mathbb{F}^n} \left[ (U_z + U_{y+z}) \cap (U_w + U_{y+w}) \subset \mathcal{L}'_z(y + z) + \mathcal{L}'_y(w) + \mathcal{L}'(y + w) \right] \geq \frac{1}{2}.
\]

Consider the most popular quadruple \( \mathcal{L}_1', \mathcal{L}_2', \mathcal{L}_3', \mathcal{L}_4' \subset \mathcal{L}' \) so that

\[
\Pr_{y, z, w \in \mathbb{F}^n} \left[ (U_z + U_{y+z}) \cap (U_w + U_{y+w}) \subset \mathcal{L}'_1(y + z) + \mathcal{L}'_2(w) + \mathcal{L}'(y + w) \right] \geq \frac{1}{2} \cdot \left( \frac{t}{d} \right)^{-4}.
\]

Let \( \mathcal{L} := \mathcal{L}_1' \cup \mathcal{L}_2' \cup \mathcal{L}_3' \cup \mathcal{L}_4' \). Recall that \( t = \exp(O(d^4)) \) and hence \( (d^4) = \exp(O(d^5)) \). We have

\[
\Pr_{y, z, w \in \mathbb{F}^n} \left[ (U_z + U_{y+z}) \cap (U_w + U_{y+w}) \subset \mathcal{L}(y + z) + \mathcal{L}(w) + \mathcal{L}(y + w) \right] \geq \exp(-O(d^5)).
\]

By averaging, there is some choice of \( z, w \) such that,

\[
\Pr_{y \in \mathbb{F}^n} \left[ (U_z + U_{y+z}) \cap (U_w + U_{y+w}) \subset \mathcal{L}(z) + \mathcal{L}(y + z) + \mathcal{L}(w) + \mathcal{L}(y + w) \right] \geq \exp(-O(d^5)).
\]

Recall that each \( L \in \mathcal{L} \) is an affine map and that \( |\mathcal{L}| \leq 4d \). Thus, \( \mathcal{L}(z), \mathcal{L}(y + z), \mathcal{L}(w), \mathcal{L}(y + w) \subset \mathcal{L}(y) + Q \) where \( Q \subset \mathbb{F}^m \) is a linear subspace of dimension \( O(d) \). We thus have

\[
B^4 := \bigcup_{y \in T} (\mathcal{L}(y) + Q) \times \{ y \} \subset A^4,
\]

where \( T \subset \mathbb{F}^n \) has density \( \exp(-O(d^5)) \).

To simplify the presentation, we would like to assume that the maps in \( \mathcal{L} \) are linear maps instead of affine maps, that is, that they do not have a constant term. This can be obtained by restricting \( x \) to the subspace orthogonal to \( Q \) and to the constant term in the affine maps in \( \mathcal{L} \). Correspondingly, we update \( r_1 := r_1 + \dim(Q) + |\mathcal{L}| = O(d) \). So from now on we may assume that \( \mathcal{L} \) is defined by \( 4d \) linear maps, and that

\[
B^4 := \bigcup_{y \in T} \mathcal{L}(y) \times \{ y \} \subset A^4,
\]

where \( T \subset \mathbb{F}^n \) has density \( \exp(-O(d^5)) \).

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Step 5. Consider $A^5 := \phi_{xy}(B^4)$ so that

$$A^5 = \bigcup_{y_1,y_2,y_3,y_4 \in T} \left( \mathcal{L}(y_1) \cap \mathcal{L}(y_2) \cap \mathcal{L}(y_3) \cap \mathcal{L}(y_4) \right) \times \{ y_1 + y_2 - y_3 - y_4 \}.$$

By Theorem 1.1 there exists a subspace $W \subset 2T - 2T$ of codimension $O(d^{20})$. However, this time, the conclusion is not strong enough for us, and we need to use Theorem 1.4 instead. The following equivalent formulation of Theorem 1.4 will be more convenient for us: there is a subspace $W \subset \mathbb{F}^n$ of codimension $O(\log^4 \alpha^{-1})$ such that, for each $y \in W$ there is a set $S_y \subset (\mathbb{F}^n)^3$ of density $\alpha^{O(1)}$, such that for all $(a_1, a_2, a_3) \in S_y$,

$$a_1 + a_2 + a_3 + a_2 - a_3 - y \in A.$$

Apply Theorem 1.4 to the set $T$ to obtain the subspace $W$ and the sets $S_y$. We have

$$B^5 := \bigcup_{y \in W} \left( \bigcup_{(y_1,y_2,y_3) \in S_y} \left( \mathcal{L}(y_1) + \mathcal{L}(y_2) + \mathcal{L}(y_3) + \mathcal{L}(y_1 + y_2 - y_3 - y) \right) \right) \times \{ y \} \subset A^5.$$

To simplify the presentation we introduce the notation $\mathcal{L}(y_1, y_2, y_3) := \mathcal{L}(y_1) + \mathcal{L}(y_2) + \mathcal{L}(y_3)$. Next, observe that for any $y, y' \in \mathbb{F}^n$, $\mathcal{L}(y) + \mathcal{L}(y') = \mathcal{L}(y') + \mathcal{L}(y)$. Thus we can simplify the expression of $B^5$ to

$$B^5 = \bigcup_{y \in W} \left( \bigcup_{(y_1,y_2,y_3) \in S_y} \left( \mathcal{L}(y_1, y_2, y_3) \cap \mathcal{L}(y) \right) \right) \times \{ y \},$$

which can be re-written as

$$B^5 = \bigcup_{y \in W} \left( \bigcup_{(y_1,y_2,y_3) \in S_y} \mathcal{L}(y_1, y_2, y_3) \cap \mathcal{L}(y) \right) \times \{ y \}.$$

Step 6. Consider $A^6 := \phi_{xy}(B^5)$. It satisfies

$$A^6 = \bigcup_{y \in W} \left( \bigcup_{(y_1,y_2,y_3) \in S_y} \left( \mathcal{L}(y_1, y_2, y_3) \cap \mathcal{L}(y) \right) \right) \times \{ y \}.$$

In order to complete the proof, we will find a large subspace $V$ such that for every $y \in W$,

$$V \cap \mathcal{L}(y) \subset \bigcup_{(y_1,y_2,y_3) \in S_y} \left( \mathcal{L}(y_1, y_2, y_3) \cap \mathcal{L}(y) \right) \cap \mathcal{L}(y) \cap \mathcal{L}(y) \cap \mathcal{L}(y).$$

In fact, we will prove something stronger: there is a large subspace $V$ such that for each $y \in W$, there is a choice of $(y_1,y_2,y_3), (y'_1,y'_2,y'_3) \in S_y$ for which

$$V \cap \mathcal{L}(y) \subset \mathcal{L}(y_1, y_2, y_3) \cap \mathcal{L}(y) \cap \mathcal{L}(y) \cap \mathcal{L}(y).$$

The following lemma is key. Given a set $\mathcal{L}$ of linear maps from $\mathbb{F}^m$ to $\mathbb{F}^n$, let $\dim(\mathcal{L})$ denote the dimension of linear span of $\mathcal{L}$ as a vector space over $\mathbb{F}$. 
Lemma 2.5. Fix $\delta > 0$. Let $\mathcal{L}$ be a set of linear maps from $\mathbb{F}^n$ to $\mathbb{F}^m$ with $\dim(\mathcal{L}) = k$. Then there is a subspace $Z \subset \mathbb{F}^m$ of dimension at most $k(2k + \log \delta^{-1} + 3)$ such that the following holds. For every subset $S \subset \mathbb{F}^n$ of density at least $\delta$, and arbitrary $y \in \mathbb{F}^n$, at least half the pairs $s, s' \in S$ satisfy

$$(\mathcal{L}(s) + \mathcal{L}(y)) \cap (\mathcal{L}(s') + \mathcal{L}(y)) \subset Z + \mathcal{L}(y).$$

Proof. The proof is by induction on $\dim(\mathcal{L})$. Consider first the base case of $\dim(\mathcal{L}) = 1$ and suppose that $\mathcal{L} = \langle L \rangle$ for some map $L$. We consider two cases based on the minimum rank of the maps in $\mathcal{L}$. First suppose that rank of every non-zero map in $\mathcal{L}$ (which is the same as rank of $L$) is bigger than $\log \delta^{-1} + 5$. Fix arbitrary $L_1, L_3 \in \mathcal{L} \setminus \{0\}$ and $L_2, L_4 \in \mathcal{L}$ and $s, y \in \mathbb{F}^n$ and observe that

$$\Pr_{s' \in S}[L_1(s) + L_2(y) = L_3(s') + L_4(y)] \leq \frac{|\mathbb{F}|^{-\log \delta^{-1} + 5}}{\Pr_{s' \in \mathbb{F}^n}[s' \in S]} \leq |\mathbb{F}|^{-\log \delta^{-1} + 5} \delta^{-1}.$$

By applying the union bound over all quadruples $L_1, \ldots, L_4 \in \mathcal{L}$, we obtain that

$$\Pr_{s, s' \in S}[(\mathcal{L}(s) + \mathcal{L}(y)) \cap (\mathcal{L}(s') + \mathcal{L}(y)) \neq \mathcal{L}(y)] \leq |\mathbb{F}|^{4} |\mathbb{F}|^{-\log \delta^{-1} + 5} \delta^{-1} \leq \frac{1}{2}.$$

Therefore, we can safely choose $Z = \{0\}$ in the lemma. Now, for the second case, suppose that $\text{rank}(L) \leq \log \delta^{-1} + 5$. Let $Z = \text{Im}(L)$. Then for all $s \in \mathbb{F}^n$, $\mathcal{L}(s) \subset \text{Im}(L) = Z$, and so $(\mathcal{L}(s) + \mathcal{L}(y)) \cap (\mathcal{L}(s') + \mathcal{L}(y)) \subset Z \subset Z + \mathcal{L}(y)$.

Now let $\dim(\mathcal{L}) = k$. First, suppose that $\forall L \in \mathcal{L}$, $\text{rank}(L) > 4k + \log \delta^{-1} + 1$. Then similar to the base case, for all $y \in \mathbb{F}^n$,

$$\Pr_{s, s' \in S}[(\mathcal{L}(s) + \mathcal{L}(y)) \cap (\mathcal{L}(s') + \mathcal{L}(y)) \neq \mathcal{L}(y)] \leq |\mathbb{F}|^{4k} |\mathbb{F}|^{-\log \delta^{-1} + 1} \delta^{-1} \leq \frac{1}{2}.$$

Otherwise, suppose there is some $L \in \mathcal{L} \setminus \{0\}$ with rank at most $4k + \log \delta^{-1} + 1$. Let $Y$ be a subspace so that $Y \oplus \text{Im}(L) = \mathbb{F}^m$. Let $\text{Proj}_Y : \mathbb{F}^n \to Y$ be the projection map along $\text{Im}(L)$ with $\text{Proj}_Y(\text{Im}(L)) = 0$. Consider the new family of maps

$$\mathcal{L}' = \{ \text{Proj}_Y \circ M : M \in \mathcal{L} \}.$$

Note that $\mathcal{L}'$ has dimension $\leq k - 1$ because $\text{Proj}_Y \circ L \equiv 0$ and so by the induction hypothesis, there exists a subspace $Z'$ of dimension at most $(k - 1)(2(k - 1) + \log \delta^{-1} + 3)$ such that, for all $y \in \mathbb{F}^n$, for least half the pairs $s, s' \in S$ it holds that

$$(\mathcal{L}'(s) + \mathcal{L}'(y)) \cap (\mathcal{L}'(s') + \mathcal{L}'(y)) \subset Z' + \mathcal{L}'(y).$$

The above implies that

$$\text{Proj}_Y((\mathcal{L}(s) + \mathcal{L}(y)) \cap (\mathcal{L}(s') + \mathcal{L}(y))) \subset Z' + \text{Proj}_Y(\mathcal{L}(y)) \subset Z' + \mathcal{L}(y) + \text{Im}(L),$$

so we can take $Z = Z' + \text{Im}(L)$. \hfill \square

We note that for Theorem 1.3 we only need a weaker form of Lemma 2.5, which states that at least one pair $s, s' \in S$ exists; however, we will need the stronger version for Theorem 1.5.

We apply Lemma 2.5 as follows. Define a new family of linear maps $\mathcal{L}^*$ from $\mathbb{F}^{3n}$ to $\mathbb{F}^m$ as follows. For each $L \in \mathcal{L}$ define three linear maps $L_i, i \in \{1, 2, 3\}$ by:

$$L_i : (\mathbb{F}^n)^3 \to \mathbb{F}^m, L_i(y_1, y_2, y_3) = L(y_i)$$

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and let
\[ \mathcal{L}^* := \{ L_i : L \in \mathcal{L}, i \in [3] \}. \]

Apply Lemma 2.5 to the family \( \mathcal{L}^* \) with \( \delta = \exp(-O(d^5)) \) and obtain a subspace \( V \subset \mathbb{P}^m \) of codimension \( O(d^5 \log(\exp(-O(d^5)))) = O(d^7) \) so that, for every \( S_y \subset (\mathbb{P}^m)^3 \) with \( y \in W \), there exist \( (y_1, y_2, y_3), (y'_1, y'_2, y'_3) \in S_y \) for which

\[ V \cap \overline{\mathcal{L}^*}(y, y, y)^\perp \subset (\overline{\mathcal{L}^*}(y_1, y_2, y_3))^\perp \cap \overline{\mathcal{L}^*}(y, y, y)^\perp + (\overline{\mathcal{L}^*}(y'_1, y'_2, y'_3))^\perp \cap \overline{\mathcal{L}^*}(y, y, y)^\perp. \]

This directly implies that

\[ V \cap \overline{\mathcal{L}}(y)^\perp \subset (\overline{\mathcal{L}}(y_1, y_2, y_3))^\perp \cap \overline{\mathcal{L}}(y)^\perp + (\overline{\mathcal{L}}(y'_1, y'_2, y'_3))^\perp \cap \overline{\mathcal{L}}(y)^\perp. \]

Define
\[ B^6 := \bigcup_{y \in W} \left( V \cap \overline{\mathcal{L}}(y)^\perp \right) \times \{ y \} \subset A^6. \]

Observe that \( B^6 \) is a bilinear variety defined by codim(\( V \)) many linear equations on \( x \), codim(\( W \)) linear equations on \( y \) and \( |\mathcal{L}| \) bilinear equations on \( (x, y) \).

To complete the proof we calculate the quantitative bounds obtained. We have \( d = O(\log^4 \alpha^{-1}) \), where \( \alpha \) was the density of the original set \( A \), and

\[ r_1 = O(d) + \text{codim}(\mathcal{L}) = O(d^7), \]

\[ r_2 = O(d) + \text{codim}(W) = O(d^{20}), \]

\[ r_3 = |\mathcal{L}| = O(d). \]

Together these give the final bound of \( r = r_1 + r_2 + r_3 = O(\log^{80} \alpha^{-1}) \).

### 3 Proof of Theorem 1.5

In this section we prove Theorem 1.5 by slightly modifying the proof of Theorem 1.3. We point out the necessary modifications to the proof of Theorem 1.3.

**Step 1.** In this step, we use Theorem 1.4 instead of Theorem 1.1 and directly obtain
\[ B^1 \subset \phi_{hh}^{\varepsilon_1}(A) \] (3)
for \( \varepsilon_1 = \alpha^{O(1)} \).

**Step 2.** Similarly in this step as well, using Theorem 1.4 instead of Theorem 1.1 gives
\[ B^2 \subset \phi_{vv}^{\varepsilon_2}(B^1) \] (4)
with \( \varepsilon_2 = \alpha^{O(1)} \). Recall that from now on we assume for simplicity of exposition that \( B^2 = \bigcup_{y \in \mathbb{P}^m} V_y \times \{ y \} \).
Steps 3 and 4. This step is slightly different from Steps 1 and 2. Here, we are not able to directly produce some set $B^i$ that would satisfy $B^i \subset \phi_n(B^2)$. But what we can do is to apply the remaining operation $\phi_{hvvh}$ to $B^2$ and obtain the final bilinear structure $B^6$ that satisfies what we want, which is

$$B^6 \subset \phi_{hvvh}(B^2)$$

(5)

for $\epsilon_6 = \exp(-\poly \log \alpha^{-1})$. Combining Equations (3) to (5) gives

$$B^6 \subset \phi_{hvvhvh}(A)$$

for $\epsilon = \exp(-\poly \log \alpha^{-1})$.

We establish Equation (5) in the rest of the proof. Recall that previously we showed that the following holds: there is a set of affine maps $\mathcal{L}$, with $|\mathcal{L}| = O(d)$, such that

$$\Pr_{y,w,z \in \mathbb{F}^n\left[(\overline{\mathcal{L}}(z) + \overline{\mathcal{L}}(y+z) + \overline{\mathcal{L}}(w) + \overline{\mathcal{L}}(y+w)) \perp \left(V_{z}^\perp \cap V_{y+z}^\perp \right) + \left(V_{w}^\perp \cap V_{y+w}^\perp \right) \right] \geq \exp(-O(d^5))$$

and consequently

$$\Pr_{y,w,z \in \mathbb{F}^n\left[(\overline{\mathcal{L}}(y) + \overline{\mathcal{L}}(z) + \overline{\mathcal{L}}(w)) \perp \left(V_{z}^\perp \cap V_{y+z}^\perp \right) + \left(V_{w}^\perp \cap V_{y+w}^\perp \right) \right] \geq \exp(-O(d^5)).$$

Recall that $d = O(\log^4 \alpha^{-1})$. Furthermore, we may assume the maps in $\mathcal{L}$ are linear (instead of affine) after we update $r_1 := r_1 + |\mathcal{L}| = O(d)$.

In the proof of Theorem 1.3 we then fixed one popular choice of $w,z$. However, here we cannot do this, as we need many pairs $w,z$. Let $T$ be the set of $y$s that satisfy

$$\Pr_{y,z \in \mathbb{F}^n\left[(\overline{\mathcal{L}}(y) + \overline{\mathcal{L}}(z) + \overline{\mathcal{L}}(w)) \perp \left(V_{z}^\perp \cap V_{y+z}^\perp \right) + \left(V_{w}^\perp \cap V_{y+w}^\perp \right) \right] \geq \exp(-O(d^5)),}$$

(6)

so $T$ has density $\exp(-O(d^5))$. We deduce something stronger from Equation (6) but we need to introduce some notation first.

For $A,B \subset \mathbb{F}^n$ let $A - \eta B$ denote the set of all elements $c \in A - B$ that can be written in at least $\eta |\mathbb{F}^n|$ many ways as $c = a - b$ for $a \in A, b \in B$. To use this notation, note that if $A,B$ are two subspaces of codimension $k$, then $A - B = A - \eta B$ for $\eta = \exp(-O(k))$. This is because every element $c \in A - B$ can be written as $c = (a + v) - (b + v)$ where $v$ is an arbitrary element in the subspace $A \cap B$ of codimension at most $2k$. So we can improve Equation (6) to

$$\Pr_{y,z \in \mathbb{F}^n\left[(\overline{\mathcal{L}}(y) + \overline{\mathcal{L}}(z) + \overline{\mathcal{L}}(w)) \perp \left(V_{z}^\perp \cap V_{y+z}^\perp \right) - \eta \left(V_{w}^\perp \cap V_{y+w}^\perp \right) \right] \geq \exp(-O(d^5)),}$$

(7)

for $\eta = \exp(-O(d))$.

Step 5. Similar to before, consider the subspace $W \subset 2T - 2T$ of codimension $O(d^{20})$ that is given by Theorem 1.4. This subspace $W$ has the following property: if we fix an arbitrary $y \in W$, sample $y_1, y_2, y_3 \in \mathbb{F}^n$ uniformly and independently, and set $y_4 = -y + y_1 + y_2 - y_3$, then with probability at least $\exp(-O(d^5))$ we have $y_1, y_2, y_3, y_4 \in T$. This means that if we furthermore sample $w_1, w_2, w_3, w_4, z_1, z_2, z_3, z_4 \in \mathbb{F}^n$ uniformly and independently, then, with probability at least $\exp(-O(d^5))$, the following four equations simultaneously hold:

$$(\overline{\mathcal{L}}(y_i) + \overline{\mathcal{L}}(z_i) + \overline{\mathcal{L}}(w_i)) \perp \left(V_{z_i}^\perp \cap V_{y_i+z_i}^\perp \right) - \eta \left(V_{w_i}^\perp \cap V_{y_i+w_i}^\perp \right) i = 1, \ldots, 4.$
By computing the intersection of the left-hand side and the right-hand side for each \( i = 1, \ldots, 4 \), we obtain that with probability at least \( \exp(-O(d^5)) \),
\[
\left( \mathcal{L}(y) + \sum_{i=1}^{3} \mathcal{L}(y_i) + \sum_{i=1}^{4} \mathcal{L}(z_i) + \sum_{i=1}^{4} \mathcal{L}(w_i) \right) \perp \subset \bigcap_{i=1}^{4} \left( \left( V_{z_i}^\perp \cap V_{y_i+z_i}^\perp \right) - \eta \left( V_{w_i}^\perp \cap V_{y_i+w_i}^\perp \right) \right).
\]
(8)

For a given \( y \in \mathbb{F}^n, s = (y_1, y_2, y_3, w_1, w_2, w_3, w_4, z_1, z_2, z_3, z_4) \in (\mathbb{F}^n)^{11} \), let
\[
\mathcal{V}_{y,s} = \bigcap_{i=1}^{4} \left( \left( V_{z_i}^\perp \cap V_{y_i+z_i}^\perp \right) - \eta \left( V_{w_i}^\perp \cap V_{y_i+w_i}^\perp \right) \right),
\]
where we recall that \( y_4 = -y + y_1 + y_2 - y_3 \). Observe that for any \( s \),
\[
\bigcup_{y \in W} \mathcal{V}_{y,s} \times \{y\} \subset \phi_{vvhv}(B^2).
\]
We rewrite Equation (8) more compactly as
\[
\Pr_s \left[ \left( \mathcal{L}(y) + \mathcal{L}(s) \right) \perp \subset \mathcal{V}_{y,s} \right] \geq \exp(-O(d^5)),
\]
where we use the notation \( \mathcal{L}(s) = \sum_{i=1}^{3} \mathcal{L}(y_i) + \sum_{i=1}^{4} \mathcal{L}(z_i) + \sum_{i=1}^{4} \mathcal{L}(w_i) \).

**Step 6.** Now we consider the ultimate result of applying the operation hvhvh to \( B^2 \). Only the final operation \( h \) remains to be applied. After doing so, we find a subspace \( V \subset \mathbb{F}^m \) of codimension \( O(d^7) \) that satisfies the following: for any \( y \in W \), choose \( s_1, s_2 \in (\mathbb{F}^n)^{11} \) uniformly and independently at random. Then with probability \( \exp(-O(d^5)) \),
\[
V \cap \mathcal{L}(y)^\perp \subset \mathcal{V}_{y,s_1} - \eta \mathcal{V}_{y,s_2},
\]
where we recall that \( \eta = \exp(-O(d)). \)

In order to determine \( V \), fix \( y \in W \) and let \( S_y \) be the set of all tuples \( s = (y_1, y_2, y_3, w_1, w_2, w_3, w_4, z_1, z_2, z_3, z_4) \in (\mathbb{F}^n)^{11} \) that satisfy Equation (9). Note that the density of each \( S_y \) is at least \( \exp(-O(d^5)) \). To simplify the notation, denote \( s = (s_1, \ldots, s_{11}) \). We invoke Lemma 2.5 in a similar way to before. Define a family \( \mathcal{L}^s \) of linear maps, containing linear maps \( L_i \) for each \( L \in \mathcal{L} \) and \( i = 1, \ldots, 11 \), where
\[
L_i : (\mathbb{F}^m)^{11} \rightarrow \mathbb{F}^m, L_i(s) = L(s_i).
\]
Applying Lemma 2.5 to \( \mathcal{L}^s \) and density parameter \( d = \exp(-O(d^5)) \), we obtain a subspace \( V \subset \mathbb{F}^m \) of codimension \( O(d^7) \) such that for each \( y \in W \),
\[
\Pr_{s_1, s_2 \in S_y} \left[ V \cap \mathcal{L}(y)^\perp \subset (\mathcal{L}(s_1) + \mathcal{L}(y))^\perp + (\mathcal{L}(s_2) + \mathcal{L}(y))^\perp \right] \geq \frac{1}{2},
\]
(10)
which implies
\[
\Pr_{s_1, s_2 \in (\mathbb{F}^n)^{11}} \left[ V \cap \mathcal{L}(y)^\perp \subset \mathcal{V}_{y,s_1} - \eta \mathcal{V}_{y,s_2} \right] \geq \exp(-O(d^5)).
\]
(11)
Define the final bilinear structure as
\[
B^5 := \bigcup_{y \in W} \left( V \cap \mathcal{L}(y)^\perp \right) \times \{y\}.
\]
It satisfies
\[ B^6 \subset \phi_{\text{hvvhv}}^{\varepsilon_6}(B^2) \]
with \( \varepsilon_6 = \exp(-O(d^5)) \), and so overall
\[ B^6 \subset \phi_{\text{hvvhvvhv}}^{\varepsilon}(A) \]
with \( \varepsilon = \exp(-O(d^5)) \).

References


